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STABILITY AND OPTIMUM DESIGN OF ARCH-TYPE STRUCTURES

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Abstract—Based on the Euler-Bernoulli Theory of the nonlinear inextensible plane elasticae, the concept of funicular or momentless design for arches is reviewed and equations determining the buckling load and transitional bending moments are derived. The results obtained for the buckling load of perabolic arches differs considerably from those found in the literature. Finally, the shape and the variable cross-sectional area of a funicular arch of constant stress is determined.

NOTATION

x, y	spatial Catesian position of arch center line
s	arch length along arch
$\mathbf{i}, \mathbf{j}$	unit Cartesian vectors
$\boldsymbol{\tau}, \mathbf{n}$	unit tangent and unit normal vectors to arc center line
T, N	axial and transverse forces in arch
$\mathbf{r}$	position vector of a point on arch center line
$\mathbf{R}$	vector of internal forces
M	bending moment
$k, \hat{k}$	curvature at stressed and unstressed states respectively
θ	inclination angle of τ with the x-axis
$f = f_x\mathbf{i} + f_y\mathbf{j}$	vector of applied external force per unit length of arch center line
p, q	applied external force per unit arc length normal and parallel to arch
h	height of symmetric arch
b	horizontal span of symmetric arches
H	maximum height of fill over buried arches
l	total arc length of arch
w	gravity loads on arches
P_0	axial compressive force at arch apex
γ	specific weight of fill on buried arches
u, v	components of displacements of arch center line
ψ	variation in θ due to deflection from funicular state
EI	bending resistance of arch
ΔT	variation in T due to deflection from funicular state
Δw	variation in the load w
Z	$T\psi - (EI\psi_s)_s$
ξ	x/b
η	y/h
g	η_ξ^2
α	$(h/b)^2$
λ	$P_0 b^2/EI$
σ_0	allowable normal stress
γ'	specific weight of arch material
w'	gravity load on arch per unit horizontal direction
β	$2\gamma' h/\sigma_0$

INTRODUCTION

Arches, as structural members, provide esthetics and strength through their geometry. In recent years their use has increased in a variety of sub-soil structures such as long span culverts and grade separations[1]. They continue to provide some of the most handsome forms of bridge design while offering at the same time greater resistance and safety.

Development of analytical methods for arches has evolved from statically determinate two and three hinged arches of an earlier time to the extensive investigations of Whitney[2] and Leontiev[3] who provide equations for determination of bending and axial stresses in arches of given geometry. Timoshenko and Young[4] point out the advantages of funicular design in which an arch geometry is selected such that under permanent design loads, the arch remains in a monentless state and experiences only axial stresses. This configuration is dependent upon the load and renders the problem nonlinear.

A momentless arch experiences bending stresses during the construction, when the design load is only partially imposed, or during service when live loads are present. Another important consideration is the question of buckling of funicular arches, which due to lack of bending stresses, may occur without noticeable prior deflection or extensive concrete cracking.

These are topics that are considered in this paper within the framework of the Euler–Bernoulli Theory of the nonlinear inextensible plane elasticae. After reviewing the theory for the determination of funicular geometry, we develop the equations governing the bending stress that may arise due to live load or partially imposed dead load. These equations as well as the equations for the buckling load of the arch are obtained by superposition of small deformations on the momentless state. This technique is standard in nonlinear elasticity and a justifiable approximation in view of the fact that live loads on such structures are generally small compared to the dead load.

We obtain the buckling loads of the parabolic arches and buried culverts supporting the weight of a variable height of soil fill. In the case of parabolic arches, our result, verified also by a perturbation analysis, differes considerably from the buckling load obtained by Dinnik[5] and reported in [6].

Finally we consider the configuration of the optimum funicular arch. This momentless arch of variable cross-section has constant axial stress and supports its own weight and the load due to the superstructure. It is shown that in the case of uniform-superstructure-load, the arch geometry remains unaffected by the magnitude of load, while its cross-sectional area is directly proportional to it. In contrast to domes of constant stresses, in which large wall thickness near the base make them esthetically undesirable, it is possible to obtain uniform depth for plane arches by increasing arch width to provide the required area.

Similar questions have continued to attract the attention of investigators. Problems of dynamic or multiple load buckling of arches have been considered by Schreyer and Masur[7], Lo and Masur[8] and Plaut[9]. Amazigo[10] has obtained optimal design against snap-buckling of shallow arches. Farshad[11] and Stadler[12] have considered the questions of optimal and natural forms for arches based upon different concepts of strength, efficiency or load bearing capacity.

Billington's review[13] of the history of esthetics and mathematical development in concrete arch bridges reveals that the architects and engineers of this century found elegance of forms in the daring and slender shapes dictated by economy, efficiency and strength. Funicular design for arches, like the membrane theory for shells, provide suitable means for optimum design of arch type structures[14].

2. BASIC EQUATIONS

Referring to Fig. 1 we denote the position vector of a point P of the central line of a plane arch by

$$\mathbf{r} = \mathbf{i}x(s) + \mathbf{j}y(s) \tag{2.1}$$

in which s is the arc length. Force and moment equilibrium require

$$\mathbf{R}_s + \mathbf{f} = 0, \qquad (\ldots)_s = \frac{\mathrm{d}}{\mathrm{d}s}(\ldots) \tag{2.2}$$

$$M_s + N = 0. \tag{2.3}$$

Here M is the bending moment, N the transverse shear, $\mathbf{f}$ the externally applied load per unit arc length and $\mathbf{R}$ the internal force vector

$$\mathbf{R} = T\boldsymbol{\tau} + N\mathbf{n}. \tag{2.4}$$

The unit orthogonal vectors $\boldsymbol{\tau}$ and $\mathbf{n}$ are given by

$$\boldsymbol{\tau} = \mathbf{r}_s = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta, \mathbf{n} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \tag{2.5}$$

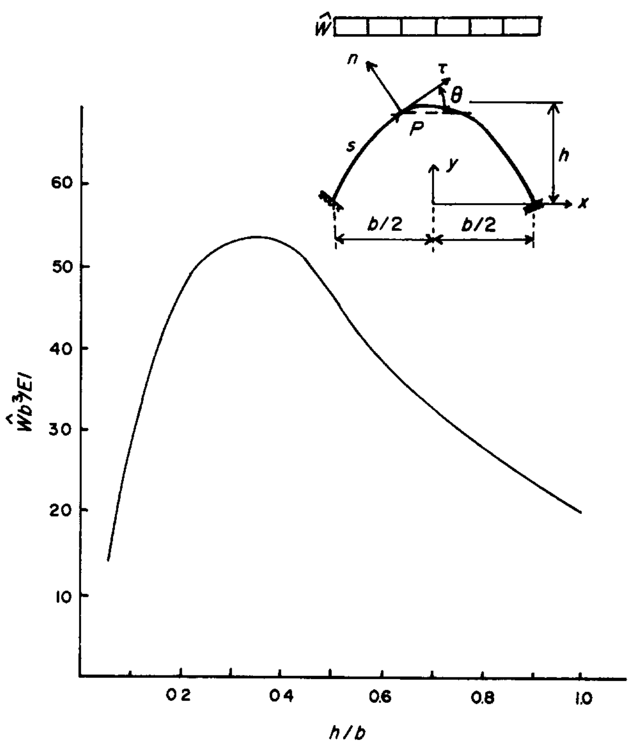

Fig. 1. Buckling load of parabolic arches.

where θ is the angle that $\boldsymbol{\tau}$ makes with the horizontal direction at P. The moment curvature relationship is expressed in the usual form

$$M = -EI(k - \hat{k}) \tag{2.6}$$

in which $\hat{k}$ is the curvature in the unstressed state and k the curvature in the stressed state given by

$$k = \mathbf{n}_s \cdot \boldsymbol{\tau} = -\boldsymbol{\tau}_s \cdot \mathbf{n} = -\theta_s. \tag{2.7}$$

The above description contains the essential attributes of the Euler–Bernoulli theory of the static inextensible plane elasticae.

Scalar multiplication of (2.2) with $\boldsymbol{\tau}$ and $\mathbf{n}$ and utilization of (2.3) and (2.7)) yields respectively

$$T_s - kM_s + q = 0 \tag{2.8}$$

$$M_{ss} + kT - p = 0 \tag{2.9}$$

where q and p are the tangential and normal components of external load

$$q = \mathbf{f} \cdot \boldsymbol{\tau}, \quad p = \mathbf{f} \cdot \mathbf{n}. \tag{2.10}$$

3. MOMENTLESS DESIGN

We shall now consider funicular configuration when the loads are only of gravitational nature. Therefore

$$\mathbf{f} = -\mathbf{j}w, \quad M = 0 \tag{3.1}$$

where w represents the total permanent design load consisting of the weight of the arch, and all other gravitational loads.

Substituting (3.1) into (2.8)–(2.9) and using (2.10) yields

$$T_s = wy_s \tag{3.2}$$

$$kT = -wx_s. \tag{3.3}$$

Dividing (3.2) by (3.3) and noting

$$(\ldots)_s = (s_x)^{-1}(\ldots)_x = (1+y_x^2)^{-1/2}(\ldots)_x$$
$$k = -(1+y_x^2)^{-3/2}y_{xx}$$

we obtain

$$T^{-1}T_x = (1+y_x^2)^{-1}y_xy_{xx}. \tag{3.4}$$

This equation can be integrated yielding

$$T = -P_0(1+y_x^2)^{1/2} \tag{3.5}$$

where P_0 is an arbitrary constant which is generally positive and for a symmetric arch is the compressive force at its apex. Equation (3.5) is valid for any variable load w.

To determine the shape of the arch, we square both sides of (3.2) and (3.3) and add, then

$$w^2 = T_s^2 + k^2T^2. \tag{3.6}$$

Substituting from (3.5) and simplyfying, (3.6) yields

$$w = \pm P_0(1+y_x^2)^{-1/2}y_{xx}. \tag{3.7}$$

For $P_0 > 0$ and for concave arches, the right choice of sign in (3.7) is negative. For any w, (3.7) determines the shape of the arch and then the axial force T can be determined from (3.5). We note here that (3.7) determines, for $w = \text{const}$, the catenary shape and parabolic arch, Fig. 1, for which $w = \hat{w}x_s$, $\hat{w} = \text{const}$.

$$y = h\left(1 - \frac{4x^2}{b^2}\right), \qquad P_0 = \frac{\hat{w}B^2}{8h}. \tag{3.8}$$

Also note that for buried culverts as shown in Fig. 2 the load due to weight of overburden is

$$w = \gamma(H-y)x_s = \gamma(H-y)(1+y_x^2)^{-1/2} \tag{3.9}$$

in which γ is the specific weight of the fill material. Equation (3.7) becomes

$$P_0y_{xx} - \gamma y = -H \tag{3.10}$$

with boundary conditions

$$y = h,\ y_x = 0 \quad x = 0; \quad \text{and} \quad y = 0, x = b/2. \tag{3.11}$$

The solution is

$$y = H - (H-h)\cosh\left(\frac{2x}{b}\cosh^{-1}\frac{H}{H-h}\right) \tag{3.12}$$

$$P_0 = \frac{4\gamma}{b^2}\left(\cosh^{-1}\frac{H}{h-h}\right)^{-2}. \tag{3.13}$$

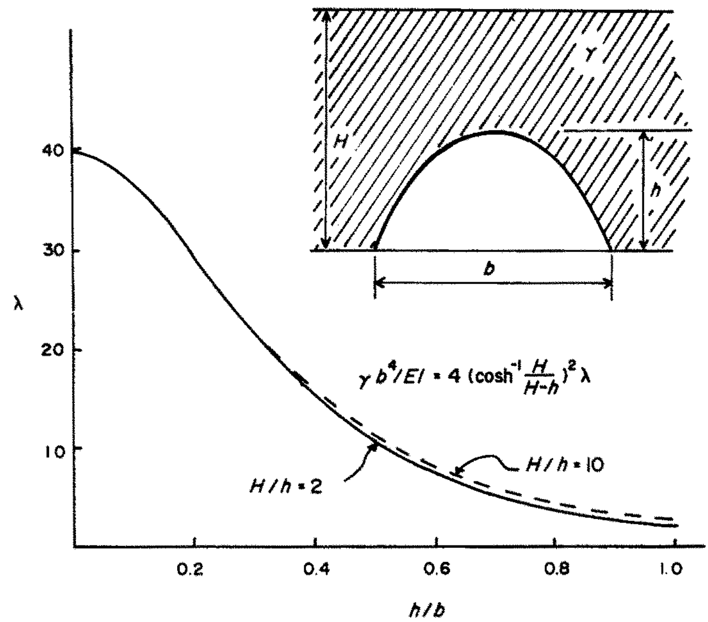

Fig. 2. Buckling load of buried arches.

4. BUCKLING OF FUNICULAR ARCHES

Let us assume that a small additional load Δw alters the funicular state characterized by equations developed in the previous section. Therefore, there will appear a bending moment M. The axial force will now be $T+\Delta T$ and the Cartesion position of a point P on the central line of the arch becomes $x+u$ and $y+v$. The condition of inextensibility, $\tau \cdot \tau = 1$, will require to linear order in the displacements u and v

$$u_s x_s + v_s y_s = 0. \tag{4.1}$$

Therefore we may assume

$$u_s = \psi y_s,\ v_s = -\psi x_s. \tag{4.2}$$

Then (4.1) is satisfied by arbitrary ψ and (2.7) determines that the variation in curvature k is given by ψ_s and therefore from (2.6)

$$M = -EI(k + \Delta k - k) = -EI\psi_s. \tag{4.3}$$

Using this result in equilibrium equations (2.8)–(2.9) and retaining linear terms in ΔT and ψ, we have

$$(\Delta T)_s + k(EI\psi_s)_s - k\Psi T = (\Delta w) y_s \tag{4.4}$$

$$k(\Delta T) - (EI\psi_s)_{ss} + (T\psi)_s = -(\Delta w) x_s. \tag{4.5}$$

Using the relation $k = -\theta_s$, $x_s = \cos\theta$, $y_s = \sin\theta$ and introducing

$$Z = T\psi - (EI\psi_s)_s \tag{4.6}$$

eqns (4.4)–(4.5) can be written in the more compact from

$$(\Delta T)_s + \theta_s Z = (\Delta w)\sin\theta \tag{4.7}$$

$$\theta_s(\Delta T) - Z_s = (\Delta w)\cos\theta. \tag{4.8}$$

The general solution of (4.7)–(4.8) can be obtained in terms of $\Delta w(s)$ and $\theta(s)$. First note that the homogeneous equations ($\Delta w = 0$) admit of the solutions $\Delta T = C\cos(\theta - \theta_0)$, $Z = C\sin(\theta - \theta_0)$ with C and θ_0 constants. By variation of parameters, it is readily established that for $\Delta w \neq 0$,

$$Z = \left(\int^s \Delta w \, ds\right)\cos\theta + C\sin(\theta - \theta_0) \tag{4.9}$$

$$\Delta T = \left(\int^s \Delta w \, ds\right)\sin\theta + C\cos(\theta - \theta_0). \tag{4.10}$$

With Z and ΔT thus found, (4.6) determines ψ when supplemented by appropriate boundary conditions.

We now turn to the question of in-plane buckling of funicular arches. Instability is indicated by the existence of a solution to the homogeneous form of (4.6). Substituting for T from (3.5) and changing independent variable to x, (4.6) becomes

$$[EI(1 + y_x^2)^{-1/2}\psi_x]_x + P_0(1 + y_x^2)\psi = 0. \tag{4.11}$$

Introducing dimensionless quantities

$$\begin{aligned} &\xi = x/b, \eta = y/h, g = \eta^2 \\ &\alpha = (h/b)^2, \lambda = P_0 b^2/EI \end{aligned} \tag{4.12}$$

(4.11) becomes

$$[(1 + \alpha g)^{-1/2}\psi_\xi]_\xi + \lambda(1 + \alpha g)\psi = 0, \qquad -1/2 < \xi < 1/2. \tag{4.13}$$

For clamped boundary conditions we have $v = v_s = 0$, which, with the aid of (4.2), imply

$$\psi = 0 \qquad \xi = \pm 1/2 \tag{4.14}$$

$$\int_{-1/2}^{1/2} \psi \, d\xi = 0. \tag{4.15}$$

Additionally we impose the normalizing condition

$$\int_{-1/2}^{1/2} \psi^2 \, d\xi = 1. \tag{4.16}$$

Using regular perturbation theory for eigenvalue problems [15] we seek solutions of the form

$$\psi(\xi, \alpha) = \sum_{n=0}^{\infty} \psi_n(\xi)\alpha^n \tag{4.17}$$

$$\lambda(\alpha) = \sum_{n=0}^{\infty} \lambda_n \alpha^n. \tag{4.18}$$

The lowest order buckling problem is governed by the differential equation

$$\psi_{0\xi\xi} + \lambda_0\psi_0 = 0 \qquad -1/2 < \xi < 1/2 \tag{4.19}$$

whose solution satisfying (4.14)–(4.16) is

$$\psi_0 = \psi_{0i} = \sqrt{(2)}\sin i\pi\left(\xi + \frac{1}{2}\right), \qquad i = 2, 4, 6, \ldots \tag{4.20}$$

$$\lambda_0 = \lambda_{0i} = i^2\pi^2, \qquad i = 2, 4, 6, \ldots \tag{4.21}$$

We note that (4.15) rules out the odd indexed eigenfunctions and eigenvalues.

The next order eigenvalue problem becomes

$$\psi_{1\xi\xi} + \lambda_0\psi_1 = \frac{1}{2}(g\psi_{0\xi})_\xi - (\lambda_0 g + \lambda_1)\psi_0, \qquad -1/2 < \xi < 1/2 \tag{4.22}$$

with ψ_1 satisfying (4.14)–(4.15) while (4.16) implies

$$\int_{-1/2}^{1/2} \psi_0\psi_1\,\mathrm{d}\xi = 0. \tag{4.23}$$

Existence of a solution to (4.22) requires that the nonhomogeneous terms on the right side be orthogonal to ψ_0. This condition determines λ_1

$$\lambda_1 = -\frac{1}{2}\int_{-1/2}^{1/2} g\psi_{0\xi}^2\,\mathrm{d}\xi - \lambda_0\int_{-1/2}^{1/2} g\psi_0^2\,\mathrm{d}\xi. \tag{4.24}$$

For parabolic arches we have, from Section 3, $P_0 = \hat{w}b\alpha^{-1/2}/8$, $g = 64\xi^2$. We find $\lambda_1 = -32\pi^2$ and note that for $a \ll 1$, $\lambda \simeq \lambda_0 + \alpha\lambda_1$ which can be written in the form

$$\frac{wb^3}{EI} \approx 8\left[\left(\frac{h}{b}\right)\lambda_0 + \left(\frac{h}{b}\right)^3\lambda_1\right]. \tag{4.25}$$

This relationship implies that wb^3/EI has a maximum at $(h/b) = (-\lambda_0/3\lambda_1)^{1/2} = 0.2042$ with the value of maximum being $(16/3)(h/b)\lambda_0 = 42.98$.

We have calculated the eigenvalue λ numerically for $0 \leq \alpha \leq 1$ by considering the Rayleigh minimum

$$\lambda = \min_{\phi} \frac{\int_{-1/2}^{1/2} (1+\alpha g)^{-1/2}\phi_\xi^2\,\mathrm{d}\xi}{\int_{-1/2}^{1/2} (1+\alpha g)\phi g)\phi^2\,\mathrm{d}\xi} \tag{4.26}$$

and using the Ritz expansion $\psi = \sum^{J} C_i\psi_{0i}$. The convergent numerical results shown in Fig. 1 indicate a maximum buckling load given by $wb^3/EI \simeq 54$ with which the two term perturbation result is in fair agreement. The result obtained by Dinnik[4] and reported in Timoshenko and Gere[5] indicates a value of 115 which is considerably larger than the values obtained here. Numerical results, based on [4.26], is also obtained for culverts and are shown in Fig. 2. Table 1 shows the typical speed of convergence of numerical results for parabolic arches.

5. OPTIMUM DESIGN OF FUNICULAR ARCHES

We define an optimum arch as a momentless arch of variable cross-section and of constant axial stress σ_0, i.e.

$$A = -T/\sigma_0. \tag{5.1}$$

Table 1. Ordinates and abscissas of the central line of the optimum arch

$\sqrt{\alpha}$ = (h/b)	λ_1 J = 10	λ_1 J = 20
0.0	0.39478424E+02	0.39477814E+02
0.1	0.36654999E+02	0.36656143E+02
0.2	0.29891235E+02	0.29892685E+02
0.3	0.22344803E+02	0.22346024E+02
0.4	0.16011673E+02	0.16013290E+02
0.6	0.81360350E+02	0.81380701E+02

The load on the arch will be assumed to consist of its weight and a variable load $w'(x)$ per unit horizontal direction, i.e.

$$w = \gamma' A + w' x_s \tag{5.2}$$

where γ' is the weight per unit volume of the arch material and $w'(x)$ represents the combined load of the road bed, superstructure, spandrels and other loads that may be considered as static.

By substitution from (3.5) and (5.1) into (5.2) we find

$$w = \frac{\gamma' P_0}{\sigma_0}(1 + y_x^2)^{1/2} + w'(1 + y_x^2)^{-1/2}. \tag{5.3}$$

Substitution of (5.3) into (3.7) yields the differential equation for the arch shape

$$y_{xx} = -\frac{\gamma'}{\sigma_0}(1 + y_x^2) - \frac{w'(x)}{P_0}. \tag{5.4}$$

Numerical integration of (5.4) can proceed as an initial value problem. For example, for a symmetric concave arch of horizontal span b and height h, we will have the initial conditions

$$y = h, \quad \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0. \tag{5.5}$$

For an assumed value of P_0, the integration can proceed until $y = 0$. The value of x at which $y = 0$ is the semi-span length $b/2$.

For the case $w' = \text{const.}$, (5.4) can be integrated explicitly. Noting that in each half-span dy/dx is a single-valued function of y, we set

$$y_x = m(y) \quad \text{implying} \quad y_{xx} = m m_y.$$

Substitution of this transformation into (5.4) leads to

$$\frac{m \, dm}{\left(\frac{w'}{P_0} + \frac{\gamma'}{\sigma_0}\right) + \frac{\gamma'}{\sigma_0} m^2} + dy = 0.$$

This equation can be integrated and, after utilization of the condition $y = h$ when $m = 0$ and reverting to original variables, leads to

$$y_x = -\left(1 + \frac{w'\sigma_0}{\gamma' P_0}\right)^{1/2} \left\{\exp\left[\frac{2\gamma'}{\sigma_0}(h - y)\right] - 1\right\}^{1/2}. \tag{5.6}$$

Separating the variables again, integrating and using the condition $y = h$ when $x = 0$, we obtain

$$\left(1 + \frac{w'\sigma_0}{\gamma' P_0}\right)^{1/2}, \quad x = \int_y^h \left\{\exp\left[\frac{2\gamma'}{\sigma_0}(h - y)\right] - 1\right\}^{-1/2} dy. \tag{5.7}$$

The condition $y = 0$ when $x = b/2$ establishes the relationship

$$\left(1 + \frac{w'\sigma_0}{\gamma' P_0}\right)^{1/2} = \frac{2}{b}\int_0^h \left\{\exp\left[\frac{2\gamma'}{\sigma_0}(h - y)\right] - 1\right\} dy. \tag{5.8}$$

Elimination of P_0 between (5.7) and (5.8) leads to the governing equation for the arch central curve

$$\int_0^y \left\{\exp\left[\frac{2\gamma'}{\sigma_0}(h - y)\right] - 1\right\}^{-1/2} dy = \left(1 - \frac{2x}{b}\right)\int_0^h \left\{\exp\left[\frac{2\gamma'}{\sigma_0}(h - y)\right] - 1\right\} dy. \tag{5.9}$$

This equation determines x as a function of y for the interval $0 \leq x \leq b/2$. It may be noted that the parameter w', the constant load per unit horizontal direcection, does not influence the arch shape.

Introducing dimensionless quantities

$$\eta = \frac{y}{h}, \xi = \frac{2x}{b}, \beta = \frac{2\gamma' h}{\sigma_0}, \alpha = \left(\frac{h}{b}\right)^2 \tag{5.10}$$

(5.9) becomes

$$\xi = 1 - \frac{F(\eta, \beta)}{F(1, \beta)} \tag{5.11}$$

where

$$F(\eta, \beta) = \int_0^\eta \{\exp[\beta(1-t)] - \}^{-1/2} \, dt. \tag{5.12}$$

It can be shown that

$$F(\eta, \beta) = 2\beta^{-1/2}(1 - \sqrt{(1-\eta)}) + 0(\beta^{1/2}). \tag{5.13}$$

Thus, in the limit $k \to 0$, (5.11) yields the parabolic shape $\xi = \sqrt{(1-\eta)}$. Tabular results for the ordinate and abscissa of the arch for several values of β is shown in Table 2.

Determination of the axial force T and area A can proceed by using (3.5) and (5.1). We note that (5.6) and (5.8) determine the slope dy/dx and the unknown parameter p_0 in terms of the known quantities γ', σ_0, h and b. The result is

$$T = -\sigma_0 A = \frac{w'\sigma_0}{\gamma'} \frac{\{4\alpha F^2(1, \beta)[e^{\beta(1-\eta)} - 1] + 1\}^{1/2}}{[4\alpha F^2(1, \beta) - 1]}. \tag{5.14}$$

We note that although the shape of the arch is unaffected by w', both T and A are directly proportional to w'. Thus the optimum arch with no load to support other than its own weight assumes zero cross-sectional area. Also (5.14) indicates that the magnitude of axial force T increases as the aspect ratio α decreases and, becomes infinite when $\alpha = 1/4F^2(1, \beta)$. Analysis of buckling strength of the optimum arch will impose a limit on the axial force. The procedure for this was indicated in Section 4 and will not be pursued further here.

Table 2. Ordinates and abscissas of the central line of the optimum arch

y/h	x/(b/2)				
	β = .050	β = .10	β = .15	β = .225	β = .275
0.0	1.00	1.00	1.00	1.00	1.00
0.1	.949	.949	.949	.949	.950
0.2	.895	.895	.896	.897	.898
0.3	.838	.839	.841	.845	.848
0.4	.776	.779	.783	.791	.796
0.5	.710	.714	.721	.732	.741
0.6	.636	.643	.652	.669	.682
0.7	.553	.562	.574	.597	.614
0.8	.454	.466	.482	.511	.533
0.9	.324	.339	.359	.395	.423
1.0	.0	.0	.0	.0	.0

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